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Lie $*$-homomorphisms between Lie C^*-algebras and Lie $*$-derivations on Lie C^*-algebras ☆

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Abstract

We prove the generalized Hyers–Ulam–Rassias stability of Lie $*$-homomorphisms in Lie C^*-algebras, and of Lie $*$-derivations on Lie C^*-algebras.

Keywords: Lie $*$-homomorphism; Lie C^*-algebra; Stability; Lie $*$-derivation

1. Introduction

Let X and Y be Banach spaces with norms $\|\cdot\|$ and $\|\cdot\|$, respectively. Consider $f : X \to Y$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. Assume that there exist constants $\theta \geqslant 0$ and $p \in [0, 1)$ such that

$$\left\| f(x+y) - f(x) - f(y) \right\| \leqslant \theta\left(\|x\|^p + \|y\|^p\right)$$

for all $x, y \in X$. Rassias [8] showed that there exists a unique $\mathbb{R}$-linear mapping $T : X \to Y$ such that

$$\left\| f(x) - T(x) \right\| \leqslant \frac{2\theta}{2 - 2^p} \|x\|^p$$

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for all $x \in X$. Găvruta [1] generalized the Rassias' result: Let G be an Abelian group and Y a Banach space. Denote by $\varphi : G \times G \to [0, \infty)$ a function such that

$$\widetilde{\varphi}(x, y) = \sum_{j=0}^{\infty} 2^{-j} \varphi\big(2^j x, 2^j y\big) < \infty$$

for all $x, y \in G$. Suppose that $f : G \to Y$ is a mapping satisfying

$$\big\| f(x+y) - f(x) - f(y) \big\| \leqslant \varphi(x, y)$$

for all $x, y \in G$. Then there exists a unique additive mapping $T : G \to Y$ such that

$$\big\| f(x) - T(x) \big\| \leqslant \frac{1}{2} \widetilde{\varphi}(x, x)$$

for all $x \in G$. Park [5] applied the Găvruta's result to linear functional equations in Banach modules over a C^*-algebra.

Jun and Lee [2] proved the following: Denote by $\varphi : X \setminus \{0\} \times X \setminus \{0\} \to [0, \infty)$ a function such that

$$\widetilde{\varphi}(x, y) = \sum_{j=0}^{\infty} 3^{-j} \varphi\big(3^j x, 3^j y\big) < \infty$$

for all $x, y \in X \setminus \{0\}$. Suppose that $f : X \to Y$ is a mapping satisfying

$$\left\| 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right\| \leqslant \varphi(x, y)$$

for all $x, y \in X \setminus \{0\}$. Then there exists a unique additive mapping $T : X \to Y$ such that

$$\big\| f(x) - f(0) - T(x) \big| \leqslant \frac{1}{3} \big(\widetilde{\varphi}(x, -x) + \widetilde{\varphi}(-x, 3x) \big)$$

for all $x \in X \setminus \{0\}$. Recently, C. Park and W. Park [7] applied the Jun and Lee's result to the Jensen's equation in Banach modules over a C^*-algebra.

Recently, Trif [9] proved the following: Let $q := l(d-1)/(d-l)$, $r := -l/(d-l)$. Denote by $\varphi : X^d \to [0, \infty)$ a function such that

$$\widetilde{\varphi}(x_1, \ldots, x_d) = \sum_{j=0}^{\infty} q^{-j} \varphi\big(q^j x_1, \ldots, q^j x_d\big) < \infty$$

for all $x_1, \ldots, x_d \in X$. Suppose that $f : X \to Y$ is a mapping satisfying

$$\begin{aligned} &\bigg\| d\, {}_{d-2}C_{l-2}\, f\left(\frac{x_1 + \cdots + x_d}{d}\right) + {}_{d-2}C_{l-1} \sum_{j=1}^{d} f(x_j) \\ &\quad - l \sum_{1 \leqslant j_1 < \cdots < j_l \leqslant d} f\left(\frac{x_{j_1} + \cdots + x_{j_l}}{l}\right) \bigg\| \leqslant \varphi(x_1, \ldots, x_d) \end{aligned}$$

for all $x_1, \ldots, x_d \in X$. Then there exists a unique additive mapping $T : X \to Y$ such that

$$\big\| f(x) - f(0) - T(x) \big\| \leqslant \frac{1}{l \cdot {}_{d-1}C_{l-1}} \widetilde{\varphi}(qx, \underbrace{rx, \ldots, rx}_{d-1 \text{ times}})$$

for all $x \in X$. And Park [6] applied the Trif's result to the Trif functional equation in Banach modules over a C^*-algebra.

A unital C^*-algebra $\mathcal{A}$, endowed with the Lie product $[x, y] = xy - yx$ on $\mathcal{A}$, is called a *Lie C^*-algebra*. A $\mathbb{C}$-linear mapping D on a Lie C^*-algebra $\mathcal{A}$ is called a *Lie derivation* if $D([x, y]) = [D(x), y] + [x, D(y)]$ holds for all $x, y \in \mathcal{A}$. A $\mathbb{C}$-linear mapping L of a Lie C^*-algebra $\mathcal{A}$ to a Lie C^*-algebra $\mathcal{B}$ is called a *Lie homomorphism* if $L([x, y]) = [L(x), L(y)]$ holds for all $x, y \in \mathcal{A}$. Throughout this paper, let $\mathcal{A}$ be a Lie C^*-algebra with norm $\|\cdot\|$ and unit e, and $\mathcal{B}$ a Lie C^*-algebra with norm $\|\cdot\|$. Let $\mathcal{U}(\mathcal{A}) = \{u \in \mathcal{A} \mid uu^* = u^*u = e\}$. Let $q = l(d-1)/(d-l)$ and $r = -l/(d-l)$ for integers l, d with $2 \leqslant l \leqslant d-1$.

In this paper, we prove the generalized Hyers–Ulam–Rassias stability of Lie $*$-homomorphisms in Lie C^*-algebras, and of Lie $*$-derivations on Lie C^*-algebras.

2. Stability of Lie $*$-homomorphisms in Lie C^*-algebras

We are going to show the generalized Hyers–Ulam–Rassias stability of Lie $*$-homomorphisms in Lie C^*-algebras associated with the Cauchy functional equation.

Theorem 2.1. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping with $h(0) = 0$ for which there exists a function $\varphi : \mathcal{A}^4 \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x, y, z, w) := \sum_{j=0}^{\infty} 2^{-j} \varphi\left(2^j x, 2^j y, 2^j z, 2^j w\right) < \infty, \tag{2.i}$$

$$\left\|h\left(\mu x + \mu y + [z, w]\right) - \mu h(x) - \mu h(y) - \left[h(z), h(w)\right]\right\| \leqslant \varphi(x, y, z, w), \tag{2.ii}$$

$$\left\|h\left(2^n u^*\right) - h\left(2^n u\right)^*\right\| \leqslant \varphi\left(2^n u, 2^n u, 0, 0\right) \tag{2.iii}$$

for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$, all $u \in \mathcal{U}(\mathcal{A})$, $n = 0, 1, \ldots,$ and all $x, y, z, w \in \mathcal{A}$. Then there exists a unique Lie $*$-homomorphism $L : \mathcal{A} \to \mathcal{B}$ such that

$$\left\|h(x) - L(x)\right\| \leqslant \frac{1}{2} \widetilde{\varphi}(x, x, 0, 0) \tag{2.iv}$$

for all $x \in \mathcal{A}$.

Proof. Put $z = w = 0$ and $\mu = 1 \in \mathbb{T}^1$ in (2.ii). It follows from Găvruta theorem [1] that there exists a unique additive mapping $L : \mathcal{A} \to \mathcal{B}$ satisfying the inequality (2.iv). The additive mapping $L : \mathcal{A} \to \mathcal{B}$ is given by

$$L(x) = \lim_{n \to \infty} \frac{1}{2^n} h\left(2^n x\right) \tag{2.1}$$

for all $x \in \mathcal{A}$.

By the assumption, for each $\mu \in \mathbb{T}^1$,

$$\left\|h\left(2^n \mu x\right) - 2\mu h\left(2^{n-1} x\right)\right\| \leqslant \varphi\left(2^{n-1} x, 2^{n-1} x, 0, 0\right)$$

for all $x \in \mathcal{A}$. And one can show that

$$\left\|\mu h\left(2^n x\right) - 2\mu h\left(2^{n-1} x\right)\right\| \leqslant |\mu| \cdot \left\|h\left(2^n x\right) - 2h\left(2^{n-1} x\right)\right\| \leqslant \varphi\left(2^{n-1} x, 2^{n-1} x, 0, 0\right)$$

for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A}$. So

$$\begin{aligned}\left\|h(2^n\mu x) - \mu h(2^n x)\right\| &\leqslant \left\|h(2^n\mu x) - 2\mu h(2^{n-1}x)\right\| + \left\|2\mu h(2^{n-1}x) - \mu h(2^n x)\right\| \\ &\leqslant \varphi(2^{n-1}x, 2^{n-1}x, 0, 0) + \varphi(2^{n-1}x, 2^{n-1}x, 0, 0)\end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A}$. Thus $2^{-n}\|h(2^n\mu x) - \mu h(2^n x)\| \to 0$ as $n \to \infty$ for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A}$. Hence

$$L(\mu x) = \lim_{n\to\infty} \frac{h(2^n\mu x)}{2^n} = \lim_{n\to\infty} \frac{\mu h(2^n x)}{2^n} = \mu L(x) \tag{2.2}$$

for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A}$.

Now let $\lambda \in \mathbb{C}$ ($\lambda \neq 0$) and M an integer greater than $4|\lambda|$. Then $|\lambda/M| < 1/4 < 1 - 2/3 = 1/3$. By [3, Theorem 1], there exist three elements $\mu_1, \mu_2, \mu_3 \in \mathbb{T}^1$ such that $3\frac{\lambda}{M} = \mu_1 + \mu_2 + \mu_3$. And $L(x) = L(3 \cdot \frac{1}{3}x) = 3L(\frac{1}{3}x)$ for all $x \in \mathcal{A}$. So $L(\frac{1}{3}x) = \frac{1}{3}L(x)$ for all $x \in \mathcal{A}$. Thus by (2.2)

$$\begin{aligned}L(\lambda x) &= L\left(\frac{M}{3} \cdot 3\frac{\lambda}{M}x\right) = M \cdot L\left(\frac{1}{3} \cdot 3\frac{\lambda}{M}x\right) = \frac{M}{3}L\left(3\frac{\lambda}{M}x\right) \\ &= \frac{M}{3}L(\mu_1 x + \mu_2 x + \mu_3 x) = \frac{M}{3}\big(L(\mu_1 x) + L(\mu_2 x) + L(\mu_3 x)\big) \\ &= \frac{M}{3}(\mu_1 + \mu_2 + \mu_3)L(x) = \frac{M}{3} \cdot 3\frac{\lambda}{M}L(x) = \lambda L(x)\end{aligned}$$

for all $x \in \mathcal{A}$. Hence

$$L(\zeta x + \eta y) = L(\zeta x) + L(\eta y) = \zeta L(x) + \eta L(y)$$

for all $\zeta, \eta \in \mathbb{C}(\zeta, \eta \neq 0)$ and all $x, y \in \mathcal{A}$. And $L(0x) = 0 = 0L(x)$ for all $x \in \mathcal{A}$. So the unique additive mapping $L : \mathcal{A} \to \mathcal{B}$ is a $\mathbb{C}$-linear mapping.

By (2.i) and (2.iii), we get

$$L(u^*) = \lim_{n\to\infty} \frac{h(2^n u^*)}{2^n} = \lim_{n\to\infty} \frac{h(2^n u)^*}{2^n} = \left(\lim_{n\to\infty} \frac{h(2^n u)}{2^n}\right)^* = L(u)^*$$

for all $u \in \mathcal{U}(\mathcal{A})$. Since $L : \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}$-linear and each $x \in \mathcal{A}$ is a finite linear combination of unitary elements (see [4, Theorem 4.1.7]), i.e., $x = \sum_{j=1}^m \lambda_j u_j$ ($\lambda_j \in \mathbb{C}, u_j \in \mathcal{U}(\mathcal{A})$),

$$\begin{aligned}L(x^*) &= L\left(\sum_{j=1}^m \bar{\lambda}_j u_j^*\right) = \sum_{j=1}^m \bar{\lambda}_j L(u_j^*) = \sum_{j=1}^m \bar{\lambda}_j L(u_j)^* = \left(\sum_{j=1}^m \lambda_j L(u_j)\right)^* \\ &= L\left(\sum_{j=1}^m \lambda_j u_j\right)^* = L(x)^*\end{aligned}$$

for all $x \in \mathcal{A}$.

It follows from (2.1) that

$$L(x) = \lim_{n\to\infty} \frac{h(2^{2n}x)}{2^{2n}} \tag{2.3}$$

for all $x \in \mathcal{A}$. Let $x = y = 0$ in (2.ii). Then we get

$$\|h([z,w]) - [h(z), h(w)]\| \leqslant \varphi(0,0,z,w)$$

for all $z, w \in \mathcal{A}$. Since

$$\frac{1}{2^{2n}}\varphi(0,0,2^n z, 2^n w) \leqslant \frac{1}{2^n}\varphi(0,0,2^n z, 2^n w),$$
$$\frac{1}{2^{2n}}\|h([2^n z, 2^n w]) - [h(2^n z), h(2^n w)]\| \leqslant \frac{1}{2^{2n}}\varphi(0,0,2^n z, 2^n w) \leqslant \frac{1}{2^n}\varphi(0,0,2^n z, 2^n w) \tag{2.4}$$

for all $z, w \in \mathcal{A}$. By (2.i), (2.3), and (2.4),

$$\begin{aligned} L([z,w]) &= \lim_{n\to\infty} \frac{h(2^{2n}[z,w])}{2^{2n}} = \lim_{n\to\infty} \frac{h([2^n z, 2^n w])}{2^{2n}} \\ &= \lim_{n\to\infty} \frac{1}{2^{2n}}[h(2^n z), h(2^n w)] = \lim_{n\to\infty}\left[\frac{h(2^n z)}{2^n}, \frac{h(2^n w)}{2^n}\right] \\ &= [L(z), L(w)] \end{aligned}$$

for all $z, w \in \mathcal{A}$. Hence the $\mathbb{C}$-linear mapping $L : \mathcal{A} \to \mathcal{B}$ is a Lie $*$-homomorphism satisfying the inequality (2.iv), as desired. □

Example 2.1. Let $\mathcal{A}$ be a unital C^*-algebra, and let a mapping $h : \mathcal{A} \to \mathcal{A}$ be defined by

$$h(x) = \begin{cases} x & \text{if } \|x\| < 1, \\ 0 & \text{if } \|x\| \geqslant 1, \end{cases}$$

for all $x \in \mathcal{A}$. Let $\varphi(x,y,z,w) = 5$ for all $x, y, z, w \in \mathcal{A}$. Then

$$\|h(\mu x + \mu y + [z,w]) - \mu h(x) - \mu h(y) - [h(z), h(w)]\| \leqslant \varphi(x,y,z,w) = 5,$$
$$\frac{1}{2}\widetilde{\varphi}(x,x,0,0) = 5 < \infty,$$
$$\|h(2^n u^*) - h(2^n u)^*\| = 0 < \varphi(2^n u, 2^n u, 0, 0) = 5$$

for all $\mu \in \mathbb{T}^1$, all $x, y, z, w \in \mathcal{A}$, $n = 0, 1, \ldots,$ and all $u \in \mathcal{U}(\mathcal{A})$. But the mapping $h : \mathcal{A} \to \mathcal{A}$ is not a Lie $*$-homomorphism.

(a) For $x = 0$,

$$L(0) = \lim_{n\to\infty} \frac{h(2^n 0)}{2^n} = \lim_{n\to\infty} \frac{h(0)}{2^n} = \lim_{n\to\infty} \frac{0}{2^n} = 0.$$

(b) For each $x \neq 0$, $\|2^n x\| = 2^n \|x\| \geqslant 1$ for all sufficiently large integers n. So

$$L(x) = \lim_{n\to\infty} \frac{h(2^n x)}{2^n} = \lim_{n\to\infty} \frac{0}{2^n} = 0.$$

Therefore, the unique Lie $*$-homomorphism $L:\mathcal{A}\to\mathcal{A}$ must be identically zero and satisfies

$$\|h(x)-L(x)\|\leqslant\frac{1}{2}\widetilde{\varphi}(x,x,0,0)=5$$

for all $x\in\mathcal{A}$.

Corollary 2.2. *Let $h:\mathcal{A}\to\mathcal{B}$ be a mapping with $h(0)=0$ for which there exist constants $\theta\geqslant 0$ and $p\in[0,1)$ such that*

$$\begin{aligned}&\big\|h\big(\mu x+\mu y+[z,w]\big)-\mu h(x)-\mu h(y)-\big[h(z),h(w)\big]\big\|\\&\quad\leqslant\theta\big(\|x\|^p+\|y\|^p+\|z\|^p+\|w\|^p\big),\\&\big\|h\big(2^n u^*\big)-h\big(2^n u\big)^*\big\|\leqslant 2\cdot 2^{np}\theta\end{aligned}$$

for all $\mu\in\mathbb{T}^1$, all $u\in\mathcal{U}(\mathcal{A})$, $n=0,1,\ldots,$ and all $x,y,z,w\in\mathcal{A}$. Then there exists a unique Lie $*$-homomorphism $L:\mathcal{A}\to\mathcal{B}$ such that

$$\|h(x)-L(x)\|\leqslant\frac{2\theta}{2-2^p}\|x\|^p$$

for all $x\in\mathcal{A}$.

Proof. Define $\varphi(x,y,z,w)=\theta(\|x\|^p+\|y\|^p+\|z\|^p+\|w\|^p)$ and apply Theorem 2.1. □

Theorem 2.3. *Let $h:\mathcal{A}\to\mathcal{B}$ be a mapping with $h(0)=0$ for which there exists a function $\varphi:\mathcal{A}^4\to[0,\infty)$ satisfying* (2.i) *and* (2.iii) *such that*

$$\big\|h\big(\mu x+\mu y+[z,w]\big)-\mu h(x)-\mu h(y)-\big[h(z),h(w)\big]\big\|\leqslant\varphi(x,y,z,w)\tag{2.v}$$

for $\mu=1,i$, and all $x,y,z,w\in\mathcal{A}$. If $h(tx)$ is continuous in $t\in\mathbb{R}$ for each fixed $x\in\mathcal{A}$, then there exists a unique Lie $*$-homomorphism $L:\mathcal{A}\to\mathcal{B}$ satisfying the inequality (2.iv).

Proof. Put $z=w=0$ and $\mu=1$ in (2.v). By the same reasoning as in the proof of Theorem 2.1, there exists a unique additive mapping $L:\mathcal{A}\to\mathcal{B}$ satisfying the inequality (2.iv). The additive mapping $L:\mathcal{A}\to\mathcal{B}$ is given by

$$L(x)=\lim_{n\to\infty}\frac{1}{2^n}h\big(2^n x\big)$$

for all $x\in\mathcal{A}$. By the same reasoning as in the proof of [8, Theorem], the additive mapping $L:\mathcal{A}\to\mathcal{B}$ is $\mathbb{R}$-linear.

Putting $y=z=w=0$ and $\mu=i$ in (2.v), we get

$$\|h(ix)-ih(x)\|\leqslant\varphi(x,0,0,0)$$

for all $x\in\mathcal{A}$. So

$$\frac{1}{2^n}\big\|h\big(2^n ix\big)-ih\big(2^n x\big)\big\|\leqslant\frac{1}{2^n}\varphi\big(2^n x,0,0,0\big),$$

which tends to zero as $n\to\infty$. Hence

$$L(ix)=\lim_{n\to\infty}\frac{h(2^n ix)}{2^n}=\lim_{n\to\infty}\frac{ih(2^n x)}{2^n}=iL(x)$$

for all $x\in\mathcal{A}$.

For each element $\lambda \in \mathbb{C}$, $\lambda = s + it$, where $s, t \in \mathbb{R}$. So

$$L(\lambda x) = L(sx + itx) = sL(x) + tL(ix) = sL(x) + itL(x) = (s + it)L(x) = \lambda L(x)$$

for all $x \in \mathcal{A}$. So

$$L(\zeta x + \eta y) = L(\zeta x) + L(\eta y) = \zeta L(x) + \eta L(y)$$

for all $\zeta, \eta \in \mathbb{C}$, and all $x, y \in \mathcal{A}$. Hence the additive mapping $L : \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}$-linear.

The rest of the proof is the same as in the proof of Theorem 2.1. □

We are going to show the generalized Hyers–Ulam–Rassias stability of Lie $*$-homomorphisms in Lie C^*-algebras associated with the Jensen functional equation.

Theorem 2.4. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping with $h(0) = 0$ for which there exists a function $\varphi : \mathcal{A}^4 \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x, y, z, w) = \sum_{j=0}^{\infty} 3^{-j} \varphi\big(3^j x, 3^j y, 3^j z, 3^j w\big) < \infty, \tag{2.vi}$$

$$\left\| 2h\left(\frac{\mu x + \mu y + [z, w]}{2}\right) - \mu h(x) - \mu h(y) - \big[h(z), h(w)\big] \right\| \leqslant \varphi(x, y, z, w), \tag{2.vii}$$

$$\big\| h\big(3^n u^*\big) - h\big(3^n u\big)^* \big\| \leqslant \varphi\big(3^n u, 3^n u, 0, 0\big) \tag{2.viii}$$

for all $\mu \in \mathbb{T}^1$, all $u \in \mathcal{U}(\mathcal{A})$, $n = 0, 1, \ldots,$ and all $x, y, z, w \in \mathcal{A} \setminus \{0\}$. Then there exists a unique Lie $*$-homomorphism $L : \mathcal{A} \to \mathcal{B}$ such that

$$\big\| h(x) - L(x) \big\| \leqslant \frac{1}{3}\big(\widetilde{\varphi}(x, -x, 0, 0) + \widetilde{\varphi}(-x, 3x, 0, 0)\big) \tag{2.ix}$$

for all $x \in \mathcal{A} \setminus \{0\}$.

Proof. Put $z = w = 0$ and $\mu = 1 \in \mathbb{T}^1$ in (2.vii). It follows from Jun and Lee theorem [2, Theorem 1] that there exists a unique additive mapping $L : \mathcal{A} \to \mathcal{B}$ satisfying the inequality (2.ix). The additive mapping $L : \mathcal{A} \to \mathcal{B}$ is given by

$$L(x) = \lim_{n \to \infty} \frac{1}{3^n} h\big(3^n x\big)$$

for all $x \in \mathcal{A}$.

By the assumption, for each $\mu \in \mathbb{T}^1$,

$$\big\| 2h\big(3^n \mu x\big) - \mu h\big(2 \cdot 3^{n-1} x\big) - \mu h\big(4 \cdot 3^{n-1} x\big) \big\| \leqslant \varphi\big(2 \cdot 3^{n-1} x, 4 \cdot 3^{n-1} x, 0, 0\big)$$

for all $x \in \mathcal{A} \setminus \{0\}$. And one can show that

$$\begin{aligned}
&\big\| \mu h\big(2 \cdot 3^{n-1} x\big) + \mu h\big(4 \cdot 3^{n-1} x\big) - 2\mu h\big(3^n x\big) \big\| \\
&\quad \leqslant |\mu| \cdot \big\| h\big(2 \cdot 3^{n-1} x\big) + h\big(4 \cdot 3^{n-1} x\big) - 2h\big(3^n x\big) \big\| \\
&\quad \leqslant \varphi\big(2 \cdot 3^{n-1} x, 4 \cdot 3^{n-1} x, 0, 0\big)
\end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A} \setminus \{0\}$. So

$$\begin{aligned}\big\|h\big(3^n\mu x\big) - \mu h\big(3^n x\big)\big\| &= \bigg\|h\big(3^n\mu x\big) - \frac{1}{2}\mu h\big(2\cdot 3^{n-1}x\big) - \frac{1}{2}\mu h\big(4\cdot 3^{n-1}x\big)\\ &\quad + \frac{1}{2}\mu h\big(2\cdot 3^{n-1}x\big) + \frac{1}{2}\mu h\big(4\cdot 3^{n-1}x\big) - \mu h\big(3^n x\big)\bigg\|\\ &\leqslant \frac{1}{2}\big\|2h\big(3^n\mu x\big) - \mu h\big(2\cdot 3^{n-1}x\big) - \mu h\big(4\cdot 3^{n-1}x\big)\big\|\\ &\quad + \frac{1}{2}\big\|\mu h\big(2\cdot 3^{n-1}x\big) + \mu h\big(4\cdot 3^{n-1}x\big) - 2\mu h\big(3^n x\big)\big\|\\ &\leqslant \frac{2}{2}\varphi\big(2\cdot 3^{n-1}x, 4\cdot 3^{n-1}x, 0, 0\big)\end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A} \setminus \{0\}$. Thus $3^{-n}\|h(3^n\mu x) - \mu h(3^n x)\| \to 0$ as $n \to \infty$ for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A} \setminus \{0\}$. Hence

$$L(\mu x) = \lim_{n\to\infty}\frac{h(3^n\mu x)}{3^n} = \lim_{n\to\infty}\frac{\mu h(3^n x)}{3^n} = \mu L(x)$$

for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A} \setminus \{0\}$.

By the same reasoning as in the proof of Theorem 2.1, the unique additive mapping $L : \mathcal{A} \to \mathcal{B}$ is a $\mathbb{C}$-linear mapping.

By a similar method to the proof of Theorem 2.1, one can show that the $\mathbb{C}$-linear mapping $L : \mathcal{A} \to \mathcal{B}$ is a Lie $*$-homomorphism satisfying the inequality (2.ix), as desired. □

Corollary 2.5. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping with $h(0) = 0$ for which there exist constants $\theta \geqslant 0$ and $p \in [0, 1)$ such that*

$$\begin{aligned}&\bigg\|2h\bigg(\frac{\mu x + \mu y + [z, w]}{2}\bigg) - \mu h(x) - \mu h(y) - \big[h(z), h(w)\big]\bigg\|\\ &\quad\leqslant \theta\big(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p\big),\\ &\big\|h\big(3^n u^*\big) - h\big(3^n u\big)^*\big\| \leqslant 2\cdot 3^{np}\theta\end{aligned}$$

for all $\mu \in \mathbb{T}^1$, all $u \in \mathcal{U}(\mathcal{A})$, $n = 0, 1, \ldots,$ and all $x, y, z, w \in \mathcal{A} \setminus \{0\}$. Then there exists a unique Lie $*$-homomorphism $L : \mathcal{A} \to \mathcal{B}$ such that

$$\big\|h(x) - L(x)\big\| \leqslant \frac{3 + 3^p}{3 - 3^p}\theta\|x\|^p$$

for all $x \in \mathcal{A} \setminus \{0\}$.

Proof. Define $\varphi(x, y, z, w) = \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)$ and apply Theorem 2.4. □

One can obtain a similar result to Theorem 2.3 for the Jensen functional equation.

Now we are going to show the generalized Hyers–Ulam–Rassias stability of Lie $*$-homomorphisms in Lie C^*-algebras associated with the Trif functional equation.

Theorem 2.6. *Let $h:\mathcal{A}\to\mathcal{B}$ be a mapping with $h(0)=0$ for which there exists a function $\varphi:\mathcal{A}^{d+2}\to[0,\infty)$ such that*

$$\widetilde{\varphi}(x_1,\dots,x_d,z,w):=\sum_{j=0}^{\infty}q^{-j}\varphi\bigl(q^jx_1,\dots,q^jx_d,q^jz,q^jw\bigr)<\infty, \tag{2.x}$$

$$\begin{aligned}&\Bigg\|{}_dC_{l-2}\,h\biggl(\frac{\mu x_1+\cdots+\mu x_d}{d}+\frac{[z,w]}{{}_{d-2}C_{l-2}}\biggr)+{}_{d-2}C_{l-1}\sum_{j=1}^{d}\mu h(x_j)\\&\quad-l\sum_{1\leqslant j_1<\cdots<j_l\leqslant d}\mu h\biggl(\frac{x_{j_1}+\cdots+x_{j_l}}{l}\biggr)-\bigl[h(z),h(w)\bigr]\Bigg\|\\&\quad\leqslant\varphi(x_1,\dots,x_d,z,w),\end{aligned} \tag{2.xi}$$

$$\bigl\|h\bigl(q^nu^*\bigr)-h\bigl(q^nu\bigr)^*\bigr\|\leqslant\varphi\bigl(\underbrace{q^nu,\dots,q^nu}_{d\ \text{times}},0,0\bigr) \tag{2.xii}$$

for all $\mu\in\mathbb{T}^1$, all $u\in\mathcal{U}(\mathcal{A})$, $n=0,1,\dots,$ and all $x_1,\dots,x_d,z,w\in\mathcal{A}$. Then there exists a unique Lie $*$-homomorphism $L:\mathcal{A}\to\mathcal{B}$ such that

$$\bigl\|h(x)-L(x)\bigr\|\leqslant\frac{1}{l\cdot{}_{d-1}C_{l-1}}\widetilde{\varphi}(qx,\underbrace{rx,\dots,rx}_{d-1\ \text{times}},0,0) \tag{2.xiii}$$

for all $x\in\mathcal{A}$.

Proof. Put $z=w=0$ and $\mu=1\in\mathbb{T}^1$ in (2.xi). It follows from Trif theorem [9, Theorem 3.1] that there exists a unique additive mapping $L:\mathcal{A}\to\mathcal{B}$ satisfying the inequality (2.xiii). The additive mapping $L:\mathcal{A}\to\mathcal{B}$ is given by

$$L(x)=\lim_{n\to\infty}\frac{1}{q^n}h\bigl(q^nx\bigr)$$

for all $x\in\mathcal{A}$.

Put $x_1=\cdots=x_d=x$ and $z=w=0$ in (2.xi). For each $\mu\in\mathbb{T}^1$,

$$\bigl\|{}_dC_{l-2}\bigl(h(\mu x)-\mu h(x)\bigr)\bigr\|\leqslant\varphi(\underbrace{x,\dots,x}_{d\ \text{times}},0,0)$$

for all $x\in\mathcal{A}$. So

$$q^{-n}\bigl\|{}_dC_{l-2}\bigl(h\bigl(\mu q^nx\bigr)-\mu h\bigl(q^nx\bigr)\bigr)\bigr\|\leqslant q^{-n}\varphi\bigl(\underbrace{q^nx,\dots,q^nx}_{d\ \text{times}},0,0\bigr)$$

for all $x\in\mathcal{A}$. By (2.x),

$$q^{-n}\bigl\|{}_dC_{l-2}\bigl(h\bigl(\mu q^nx\bigr)-\mu h\bigl(q^nx\bigr)\bigr)\bigr\|\to0$$

as $n\to\infty$ for all $\mu\in\mathbb{T}^1$ and all $x\in\mathcal{A}$. Thus

$$q^{-n}\bigl\|h\bigl(\mu q^nx\bigr)-\mu h\bigl(q^nx\bigr)\bigr\|\to0$$

as $n \to \infty$ for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A}$. Hence

$$L(\mu x) = \lim_{n\to\infty} \frac{h(q^n \mu x)}{q^n} = \lim_{n\to\infty} \frac{\mu h(q^n x)}{q^n} = \mu L(x)$$

for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A}$.

By the same reasoning as in the proof of Theorem 2.1, the unique additive mapping $L : \mathcal{A} \to \mathcal{B}$ is a $\mathbb{C}$-linear mapping.

By a similar method to the proof of Theorem 2.1, one can show that the $\mathbb{C}$-linear mapping $L : \mathcal{A} \to \mathcal{B}$ is a Lie $*$-homomorphism satisfying the inequality (2.xiii), as desired. □

Corollary 2.7. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping with $h(0) = 0$ for which there exist constants $\theta \geqslant 0$ and $p \in [0, 1)$ such that*

$$\begin{aligned}
&\Bigg\| {}_{d-2}C_{l-2}\, h\left(\frac{\mu x_1 + \cdots + \mu x_d}{d} + \frac{[z, w]}{{}_{d-2}C_{l-2}}\right) + {}_{d-2}C_{l-1} \sum_{j=1}^{d} \mu h(x_j) \\
&\qquad - l \sum_{1 \leqslant j_1 < \cdots < j_l \leqslant d} \mu h\left(\frac{x_{j_1} + \cdots + x_{j_l}}{l}\right) - \big[h(z), h(w)\big] \Bigg\| \\
&\quad \leqslant \theta \left(\sum_{j=1}^{d} \|x_j\|^p + \|z\|^p + \|w\|^p \right), \\
&\big\| h\big(q^n u^*\big) - h\big(q^n u\big)^* \big\| \leqslant d q^{np} \theta
\end{aligned}$$

for all $\mu \in \mathbb{T}^1$, all $u \in \mathcal{U}(\mathcal{A})$, $n = 0, 1, \ldots,$ and all $x_1, \ldots, x_d, z, w \in \mathcal{A}$. Then there exists a unique Lie $*$-homomorphism $L : \mathcal{A} \to \mathcal{B}$ such that

$$\big\| h(x) - L(x) \big\| \leqslant \frac{q^{1-p}(q^p + (d-1)r^p)\theta}{l\, {}_{d-1}C_{l-1}(q^{1-p} - 1)} \|x\|^p$$

for all $x \in \mathcal{A}$.

Proof. Define $\varphi(x_1, \ldots, x_d, z, w) = \theta(\sum_{j=1}^{d} \|x_j\|^p + \|z\|^p + \|w\|^p)$, and apply Theorem 2.6. □

One can obtain a similar result to Theorem 2.3 for the Trif functional equation.

3. Stability of Lie $*$-derivations on Lie C^*-algebras

We are going to show the generalized Hyers–Ulam–Rassias stability of Lie $*$-derivations on Lie C^*-algebras associated with the Cauchy functional equation.

Theorem 3.1. *Let $h : \mathcal{A} \to \mathcal{A}$ be a mapping with $h(0) = 0$ for which there exists a function $\varphi : \mathcal{A}^4 \to [0, \infty)$ satisfying* (2.i) *and* (2.iii) *such that*

$$\begin{aligned}
&\big\| h\big(\mu x + \mu y + [z, w]\big) - \mu h(x) - \mu h(y) - \big[h(z), w\big] - \big[z, h(w)\big] \big\| \\
&\quad \leqslant \varphi(x, y, z, w) \qquad\qquad (3.i)
\end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z, w \in \mathcal{A}$. Then there exists a unique Lie $*$-derivation $D : \mathcal{A} \to \mathcal{A}$ such that

$$\left\|h(x) - D(x)\right\| \leqslant \frac{1}{2}\widetilde{\varphi}(x, x, 0, 0) \tag{3.ii}$$

for all $x \in \mathcal{A}$.

Proof. Put $z = w = 0$ in (3.i). By the same reasoning as in the proof of Theorem 2.1, there exists a unique $\mathbb{C}$-linear involutive mapping $D : \mathcal{A} \to \mathcal{A}$ satisfying the inequality (3.ii). The $\mathbb{C}$-linear mapping $D : \mathcal{A} \to \mathcal{A}$ is given by

$$D(x) = \lim_{n\to\infty} \frac{1}{2^n} h\left(2^n x\right) \tag{3.1}$$

for all $x \in \mathcal{A}$.

It follows from (3.1) that

$$D(x) = \lim_{n\to\infty} \frac{h(2^{2n} x)}{2^{2n}} \tag{3.2}$$

for all $x \in \mathcal{A}$. Let $x = y = 0$ in (3.i). Then we get

$$\left\|h\left([z, w]\right) - \left[h(z), w\right] - \left[z, h(w)\right]\right\| \leqslant \varphi(0, 0, z, w)$$

for all $z, w \in \mathcal{A}$. Since

$$\begin{aligned}
&\frac{1}{2^{2n}}\varphi\left(0, 0, 2^n z, 2^n w\right) \leqslant \frac{1}{2^n}\varphi\left(0, 0, 2^n z, 2^n w\right), \\
&\frac{1}{2^{2n}}\left\|h\left(\left[2^n z, 2^n w\right]\right) - \left[h\left(2^n z\right), 2^n w\right] - \left[2^n z, h\left(2^n w\right)\right]\right\| \\
&\quad \leqslant \frac{1}{2^{2n}}\varphi\left(0, 0, 2^n z, 2^n w\right) \leqslant \frac{1}{2^n}\varphi\left(0, 0, 2^n z, 2^n w\right)
\end{aligned} \tag{3.3}$$

for all $z, w \in \mathcal{A}$. By (2.i), (3.2), and (3.3),

$$\begin{aligned}
D\left([z, w]\right) &= \lim_{n\to\infty} \frac{h(2^{2n}[z, w])}{2^{2n}} = \lim_{n\to\infty} \frac{h([2^n z, 2^n w])}{2^{2n}} \\
&= \lim_{n\to\infty}\left(\left[\frac{h(2^n z)}{2^n}, \frac{2^n w}{2^n}\right] + \left[\frac{2^n z}{2^n}, \frac{h(2^n w)}{2^n}\right]\right) \\
&= \left[D(z), w\right] + \left[z, D(w)\right]
\end{aligned}$$

for all $z, w \in \mathcal{A}$. Hence the $\mathbb{C}$-linear mapping $D : \mathcal{A} \to \mathcal{A}$ is a Lie $*$-derivation satisfying the inequality (3.ii), as desired. □

Corollary 3.2. *Let $h : \mathcal{A} \to \mathcal{A}$ be a mapping with $h(0) = 0$ for which there exist constants $\theta \geqslant 0$ and $p \in [0, 1)$ such that*

$$\begin{aligned}
&\left\|h\left(\mu x + \mu y + [z, w]\right) - \mu h(x) - \mu h(y) - \left[h(z), w\right] - \left[z, h(w)\right]\right\| \\
&\quad \leqslant \theta\left(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p\right), \\
&\left\|h\left(2^n u^*\right) - h\left(2^n u\right)^*\right\| \leqslant 2 \cdot 2^{np}\theta
\end{aligned}$$

for all $\mu \in \mathbb{T}^1$, *all* $u \in \mathcal{U}(\mathcal{A})$, $n = 0, 1, \ldots,$ *and all* $x, y, z, w \in \mathcal{A}$. *Then there exists a unique Lie* $*$*-derivation* $D : \mathcal{A} \to \mathcal{A}$ *such that*

$$\left\|h(x) - D(x)\right\| \leqslant \frac{2\theta}{2 - 2^p}\|x\|^p$$

for all $x \in \mathcal{A}$.

Proof. Define $\varphi(x, y, z, w) = \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)$ and apply Theorem 3.1. □

Theorem 3.3. *Let* $h : \mathcal{A} \to \mathcal{A}$ *be a mapping with* $h(0) = 0$ *for which there exists a function* $\varphi : \mathcal{A}^4 \to [0, \infty)$ *satisfying* (2.i) *and* (2.iii) *such that*

$$\left\|h\left(\mu x + \mu y + [z, w]\right) - \mu h(x) - \mu h(y) - \left[h(z), w\right] - \left[z, h(w)\right]\right\| \leqslant \varphi(x, y, z, w)$$

for $\mu = 1, i$, *and all* $x, y, z, w \in \mathcal{A}$. *If* $h(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in \mathcal{A}$, *then there exists a unique Lie* $*$*-derivation* $D : \mathcal{A} \to \mathcal{A}$ *satisfying the inequality* (3.ii).

Proof. By the same reasoning as in the proof of Theorem 2.3, there exists a unique $\mathbb{C}$-linear mapping $D : \mathcal{A} \to \mathcal{A}$ satisfying the inequality (3.ii).

The rest of the proof is the same as the proofs of Theorems 2.1 and 3.1. □

We are going to show the generalized Hyers–Ulam–Rassias stability of Lie $*$-derivations on Lie C^*-algebras associated with the Jensen functional equation.

Theorem 3.4. *Let* $h : \mathcal{A} \to \mathcal{A}$ *be a mapping with* $h(0) = 0$ *for which there exists a function* $\varphi : \mathcal{A}^4 \to [0, \infty)$ *satisfying* (2.vi) *and* (2.viii) *such that*

$$\begin{aligned} &\left\|2h\left(\frac{\mu x + \mu y + [z, w]}{2}\right) - \mu h(x) - \mu h(y) - \left[h(z), w\right] - \left[z, h(w)\right]\right\| \\ &\quad \leqslant \varphi(x, y, z, w) \end{aligned} \tag{3.iii}$$

for all $\mu \in \mathbb{T}^1$ *and all* $x, y, z, w \in \mathcal{A} \setminus \{0\}$. *Then there exists a unique Lie* $*$*-derivation* $D : \mathcal{A} \to \mathcal{A}$ *such that*

$$\left\|h(x) - D(x)\right\| \leqslant \frac{1}{3}\left(\widetilde{\varphi}(x, -x, 0, 0) + \widetilde{\varphi}(-x, 3x, 0, 0)\right) \tag{3.iv}$$

for all $x \in \mathcal{A} \setminus \{0\}$.

Proof. Put $z = w = 0$ in (3.iii). By the same reasoning as in the proof of Theorem 2.4, there exists a unique $\mathbb{C}$-linear involutive mapping $D : \mathcal{A} \to \mathcal{A}$ satisfying the inequality (3.iv). The $\mathbb{C}$-linear mapping $D : \mathcal{A} \to \mathcal{A}$ is given by

$$D(x) = \lim_{n\to\infty} \frac{1}{3^n} h\left(3^n x\right) \tag{3.4}$$

for all $x \in \mathcal{A}$.

It follows from (3.4) that

$$D(x) = \lim_{n\to\infty} \frac{h(3^{2n}x)}{3^{2n}} \tag{3.5}$$

for all $x \in \mathcal{A}$. Let $x = y = 0$ in (3.iii). Then we get

$$\left\| 2h\left(\frac{[z,w]}{2}\right) - \left[h(z), w\right] - \left[z, h(w)\right] \right\| \leqslant \varphi(0,0,z,w)$$

for all $z, w \in \mathcal{A}$. Since

$$\frac{1}{3^{2n}}\varphi\left(0,0,3^n z, 3^n w\right) \leqslant \frac{1}{3^n}\varphi\left(0,0,3^n z, 3^n w\right),$$

$$\frac{1}{3^{2n}}\left\| 2h\left(\frac{1}{2}\left[3^n z, 3^n w\right]\right) - \left[h\left(3^n z\right), 3^n w\right] - \left[3^n z, h\left(3^n w\right)\right] \right\|$$
$$\leqslant \frac{1}{3^{2n}}\varphi\left(0,0,3^n z, 3^n w\right) \leqslant \frac{1}{3^n}\varphi\left(0,0,3^n z, 3^n w\right) \tag{3.6}$$

for all $z, w \in \mathcal{A}$. By (2.vi), (3.5), and (3.6),

$$\begin{aligned} 2D\left(\frac{[z,w]}{2}\right) &= \lim_{n\to\infty} \frac{2h\left(\frac{3^{2n}}{2}[z,w]\right)}{3^{2n}} = \lim_{n\to\infty} \frac{2h\left(\frac{1}{2}[3^n z, 3^n w]\right)}{3^{2n}} \\ &= \lim_{n\to\infty}\left(\left[\frac{h(3^n z)}{3^n}, \frac{3^n w}{3^n}\right] + \left[\frac{3^n z}{3^n}, \frac{h(3^n w)}{3^n}\right]\right) \\ &= \left[D(z), w\right] + \left[z, D(w)\right] \end{aligned}$$

for all $z, w \in \mathcal{A}$. But since D is $\mathbb{C}$-linear,

$$D\left([z,w]\right) = 2D\left(\frac{[z,w]}{2}\right) = \left[D(z), w\right] + \left[z, D(w)\right]$$

for all $z, w \in \mathcal{A}$. Hence the $\mathbb{C}$-linear mapping $D : \mathcal{A} \to \mathcal{A}$ is a Lie $*$-derivation satisfying the inequality (3.iv), as desired. □

Corollary 3.5. *Let $h : \mathcal{A} \to \mathcal{A}$ be a mapping with $h(0) = 0$ for which there exist constants $\theta \geqslant 0$ and $p \in [0,1)$ such that*

$$\left\| 2h\left(\frac{\mu x + \mu y + [z,w]}{2}\right) - \mu h(x) - \mu h(y) - \left[h(z), w\right] - \left[z, h(w)\right] \right\|$$
$$\leqslant \theta\left(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p\right),$$
$$\left\| h\left(3^n u^*\right) - h\left(3^n u\right)^* \right\| \leqslant 2 \cdot 3^{np}\theta$$

for all $\mu \in \mathbb{T}^1$, all $u \in \mathcal{U}(\mathcal{A})$, $n = 0, 1, \ldots,$ and all $x, y, z, w \in \mathcal{A} \setminus \{0\}$. Then there exists a unique Lie $*$-derivation $D : \mathcal{A} \to \mathcal{A}$ such that

$$\left\| h(x) - D(x) \right\| \leqslant \frac{3 + 3^p}{3 - 3^p}\theta\|x\|^p$$

for all $x \in \mathcal{A} \setminus \{0\}$.

Proof. Define $\varphi(x,y,z,w) = \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)$ and apply Theorem 3.4. □

One can obtain a similar result to Theorem 3.3 for the Jensen functional equation.

Now we are going to show the generalized Hyers–Ulam–Rassias stability of Lie $*$-derivations on Lie C^*-algebras associated with the Trif functional equation.

Theorem 3.6. *Let $h : \mathcal{A} \to \mathcal{A}$ be a mapping with $h(0) = 0$ for which there exists a function $\varphi : \mathcal{A}^{d+2} \to [0, \infty)$ satisfying* (2.x) *and* (2.xii) *such that*

$$\begin{aligned} &\left\| {}_dd_{-2}C_{l-2}h\left(\frac{\mu x_1 + \cdots + \mu x_d}{d} + \frac{[z, w]}{{}_dd_{-2}C_{l-2}}\right) + {}_{d-2}C_{l-1}\sum_{j=1}^{d}\mu h(x_j)\right.\\ &\qquad \left. - l\sum_{1\leqslant j_1<\cdots<j_l\leqslant d}\mu h\left(\frac{x_{j_1} + \cdots + x_{j_l}}{l}\right) - \big[h(z), w\big] - \big[z, h(w)\big]\right\|\\ &\quad \leqslant \varphi(x_1, \ldots, x_d, z, w) \end{aligned} \tag{3.v}$$

for all $\mu \in \mathbb{T}^1$ and all $x_1, \ldots, x_d, z, w \in \mathcal{A}$. Then there exists a unique Lie $*$-derivation $D : \mathcal{A} \to \mathcal{A}$ such that

$$\big\|h(x) - D(x)\big\| \leqslant \frac{1}{l \cdot {}_{d-1}C_{l-1}}\widetilde{\varphi}(qx, \underbrace{rx, \ldots, rx}_{d-1 \text{ times}}, 0, 0) \tag{3.vi}$$

for all $x \in \mathcal{A}$.

Proof. Put $z = w = 0$ in (3.v). By the same reasoning as in the proof of Theorem 2.6, there exists a unique $\mathbb{C}$-linear involutive mapping $D : \mathcal{A} \to \mathcal{A}$ satisfying the inequality (3.vi). The $\mathbb{C}$-linear mapping $D : \mathcal{A} \to \mathcal{A}$ is given by

$$D(x) = \lim_{n\to\infty}\frac{1}{q^n}h\big(q^n x\big) \tag{3.7}$$

for all $x \in \mathcal{A}$.

It follows from (3.7) that

$$D(x) = \lim_{n\to\infty}\frac{h(q^{2n}x)}{q^{2n}} \tag{3.8}$$

for all $x \in \mathcal{A}$. Let $x_1 = \cdots = x_d = 0$ in (3.v). Then we get

$$\left\| {}_dd_{-2}C_{l-2}h\left(\frac{[z, w]}{{}_dd_{-2}C_{l-2}}\right) - \big[h(z), w\big] - \big[z, h(w)\big]\right\| \leqslant \varphi(\underbrace{0, \ldots, 0}_{d \text{ times}}, z, w)$$

for all $z, w \in \mathcal{A}$. Since

$$\begin{aligned} &\frac{1}{q^{2n}}\varphi\big(\underbrace{0, \ldots, 0}_{d \text{ times}}, q^n z, q^n w\big) \leqslant \frac{1}{q^n}\varphi\big(\underbrace{0, \ldots, 0}_{d \text{ times}}, q^n z, q^n w\big),\\ &\frac{1}{q^{2n}}\left\| {}_dd_{-2}C_{l-2}h\left(\frac{1}{{}_dd_{-2}C_{l-2}}\big[q^n z, q^n w\big]\right) - \big[h\big(q^n z\big), q^n w\big] - \big[q^n z, h\big(q^n w\big)\big]\right\|\\ &\quad \leqslant \frac{1}{q^{2n}}\varphi\big(\underbrace{0, \ldots, 0}_{d \text{ times}}, q^n z, q^n w\big) \leqslant \frac{1}{q^n}\varphi\big(\underbrace{0, \ldots, 0}_{d \text{ times}}, q^n z, q^n w\big) \end{aligned} \tag{3.9}$$

for all $z, w \in \mathcal{A}$.

By (2.x), (3.8), and (3.9),

$$\begin{aligned} d_{d-2}C_{l-2}D\left(\frac{[z,w]}{d_{d-2}C_{l-2}}\right) &= \lim_{n\to\infty}\frac{d_{d-2}C_{l-2}h\left(\frac{q^{2n}}{d_{d-2}C_{l-2}}[z,w]\right)}{q^{2n}} \\ &= \lim_{n\to\infty}\frac{d_{d-2}C_{l-2}h\left(\frac{1}{d_{d-2}C_{l-2}}[q^n z,q^n w]\right)}{q^{2n}} \\ &= \lim_{n\to\infty}\left(\left[\frac{h(q^n z)}{q^n},\frac{q^n w}{q^n}\right]+\left[\frac{q^n z}{q^n},\frac{h(q^n w)}{q^n}\right]\right) \\ &= \left[D(z),w\right]+\left[z,D(w)\right] \end{aligned}$$

for all $z, w \in \mathcal{A}$. But since D is $\mathbb{C}$-linear,

$$D\big([z,w]\big) = d_{d-2}C_{l-2}D\left(\frac{[z,w]}{d_{d-2}C_{l-2}}\right) = \left[D(z),w\right]+\left[z,D(w)\right]$$

for all $z, w \in \mathcal{A}$. Hence the $\mathbb{C}$-linear mapping $D:\mathcal{A}\to\mathcal{A}$ is a Lie $*$-derivation satisfying the inequality (3.vi), as desired. □

Corollary 3.7. *Let $h:\mathcal{A}\to\mathcal{A}$ be a mapping with $h(0)=0$ for which there exist constants $\theta \geqslant 0$ and $p\in[0,1)$ such that*

$$\begin{aligned} &\left\| d_{d-2}C_{l-2}h\left(\frac{\mu x_1+\cdots+\mu x_d}{d}+\frac{[z,w]}{d_{d-2}C_{l-2}}\right)+{}_{d-2}C_{l-1}\sum_{j=1}^{d}\mu h(x_j)\right. \\ &\qquad \left. -\, l\sum_{1\leqslant j_1<\cdots<j_l\leqslant d}\mu h\left(\frac{x_{j_1}+\cdots+x_{j_l}}{l}\right)-\left[h(z),w\right]-\left[z,h(w)\right]\right\| \\ &\quad \leqslant \theta\left(\sum_{j=1}^{d}\|x_j\|^p+\|z\|^p+\|w\|^p\right), \\ &\left\|h\big(q^n u^*\big)-h\big(q^n u\big)^*\right\| \leqslant dq^{np}\theta \end{aligned}$$

for all $\mu\in\mathbb{T}^1$, all $u\in\mathcal{U}(\mathcal{A})$, $n=0,1,\ldots,$ and all $x_1,\ldots,x_d,z,w\in\mathcal{A}$. Then there exists a unique Lie $*$-derivation $D:\mathcal{A}\to\mathcal{A}$ such that

$$\left\|h(x)-D(x)\right\| \leqslant \frac{q^{1-p}(q^p+(d-1)r^p)\theta}{l_{d-1}C_{l-1}(q^{1-p}-1)}\|x\|^p$$

for all $x\in\mathcal{A}$.

Proof. Define $\varphi(x_1,\ldots,x_d,z,w)=\theta(\sum_{j=1}^{d}\|x_j\|^p+\|z\|^p+\|w\|^p)$ and apply Theorem 3.6. □

One can obtain a similar result to Theorem 3.3 for the Trif functional equation.

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